Polynomial Interpolation

We often know the value of a function $f(x)$ at a set of n points in a range $[a, b]$ with $a = x_1 < \ x_2 < \cdots < x_n = b$ and we are required to determine an estimate of $f(x)$ at any value of $\in [a, b]$. This method of approximation is called interpolation[1]. The most common form of interpolation is to join the points by a polynomial and this is termed polynomial interpolation.

In general a set of n points $(x_i, f(x_i))$ for *i=1,2,…n* can be interpolated by a polynomial of degree *n-1*:

$$f(x) \approx p_{n-1}(x) = a_{n-1}\, x^{n-1} + \ a_{n-2}\, x^{n-2} + \ \ldots\ a_1\, x + \ a_0\,. \qquad (1)$$

The functions $f(x)$ and $p_{n-1}(x)$ match at the interpolation points; $p_{n-1}(x_i) = f(x_i)$ for $= 1,2, \ldots n$. In special cases, for example when $f(x)$ is a polynomial of degree less than *n-1*, then the polynomial interpolant has a degree equal to the degree of $f(x)$, and the function $f(x)$ and $p_{n-1}(x)$ are identical.

The polynomial $p_{n-1}(x)$ can be considered to provide an approximation of $f(x)$, within some range. If the range is within $[\, x_1, x_n]$ then the process is termed interpolation. If the technique is used to estimate a value of $f(x)$ outside the range $[\, x_1, x_n]$ then it is termed extrapolation. The polynomial interpolant than passes through a set of points can be found by a technique termed *Newton's divided difference method.*

Newton's Divided Difference Method

For a set of n points $(x_i, f(x_i))$ for *i=1,2,…n,* the Newton's divided difference method involves first forming a table of differences as follows.

x_i	$f(x_i)$	1st difference	2nd difference		
x_1	f_1			………..	
x_2	f_2	$\frac{f_2 - f_1}{x_2 - x_1} = F_{22}$		………..	
x_3	f_3	$\frac{f_3 - f_2}{x_3 - x_2} = F_{32}$	$\frac{F_{32} - F_{22}}{x_3 - x_1} = F_{33}$		
x_4	f_4	$\frac{f_4 - f_3}{x_4 - x_3} = F_{42}$	$\frac{F_{42} - F_{32}}{x_4 - x_2} = F_{43}$		
x_5	f_5	$\frac{f_5 - f_4}{x_5 - x_4} = F_{52}$	$\frac{F_{52} - F_{42}}{x_5 - x_3} = F_{53}$		
⋮	⋮	$\frac{f_6 - f_5}{x_6 - x_5} = F_{62}$	$\frac{F_{62} - F_{52}}{x_6 - x_4} = F_{63}$		
x_{n-}	f_{n-2}	⋮	⋮		
x_{n-}	f_{n-1}	$\frac{f_{n-1} - f_{n-2}}{x_{n-1} - x_{n-2}} = F_{n-1\,2}$	⋮		
x_n	f_n	$\frac{f_n - f_{n-1}}{x_n - x_{n-1}} = F_{n\,2}$	$\frac{F_{n\,2} - F_{n-1\,2}}{x_n - x_{n-2}} = F_{n\,3}$		$\frac{F_{n\,n-1} - F_{n-1\,n-1}}{x_n - x_1} = F_{n\,n}$

[1] Approximation: An introduction to interpolation and curve fitting.

The polynomial interpolant $p_{n-1}(x)$ may be defined using the diagonal of values and the values of x_i, as follows.

$$p_{n-1}(x) = f_1 + F_{22}(x - x_1) + F_{13}(x - x_1)(x - x_2) + \cdots + F_{1n}(x - x_1)(x - x_2) \dots (x - x_{n-1})$$

Example

Find the cubic interpolant that passes through the data points (0, 2), (2, 5), (3, 7) and (6,3).

x_i	$f(x_i)$	1st difference	2nd difference	3rd difference
-2	-51			
1	-3	16		
3	39	21	1	
4	117	78	19	3

Hence the cubic interpolant is

$$p_3(x) = -51 + 16(x+2) + (x+2)(x-1) + 3(x+2)(x-1)(x-3)$$

We note that the interpolant is a cubic. Just to check that the interpolant passes through the data points, we note that

$$p_3(-2) = -51 + 16(-2+2) + (-2+2)(-2-1) + 3(-2+2)(-2-1)(-2-3) = -51,$$

$$p_3(1) = -51 + 16(1+2) + (1+2)(1-1) + 3(1+2)(1-1)(1-3) = -51 + 16 \times 3 + 0 + 0 = -3$$

$$p_3(3) = -51 + 16(3+2) + (3+2)(3-1) + 3(3+2)(3-1)(3-3) = -51 + 16 \times 5 + 5 \times 2 + 0 = 39$$

$$p_3(4) = -51 + 16(4+2) + (4+2)(4-1) + 3(4+2)(4-1)(4-3) = -51 + 16 \times 6 + 6 \times 3 + 3 \times 6 \times 3 \times 1 = 117,$$

Computational issues

Normally the divided differences are centred, when they are laid out. So for example the normal layout for the divided difference above is as follows.

x_i	$f(x_i)$	1st difference	2nd difference	3rd difference
-2	-51			
		16		
1	-3		1	
		21		3
3	39		19	
		78		
4	117			

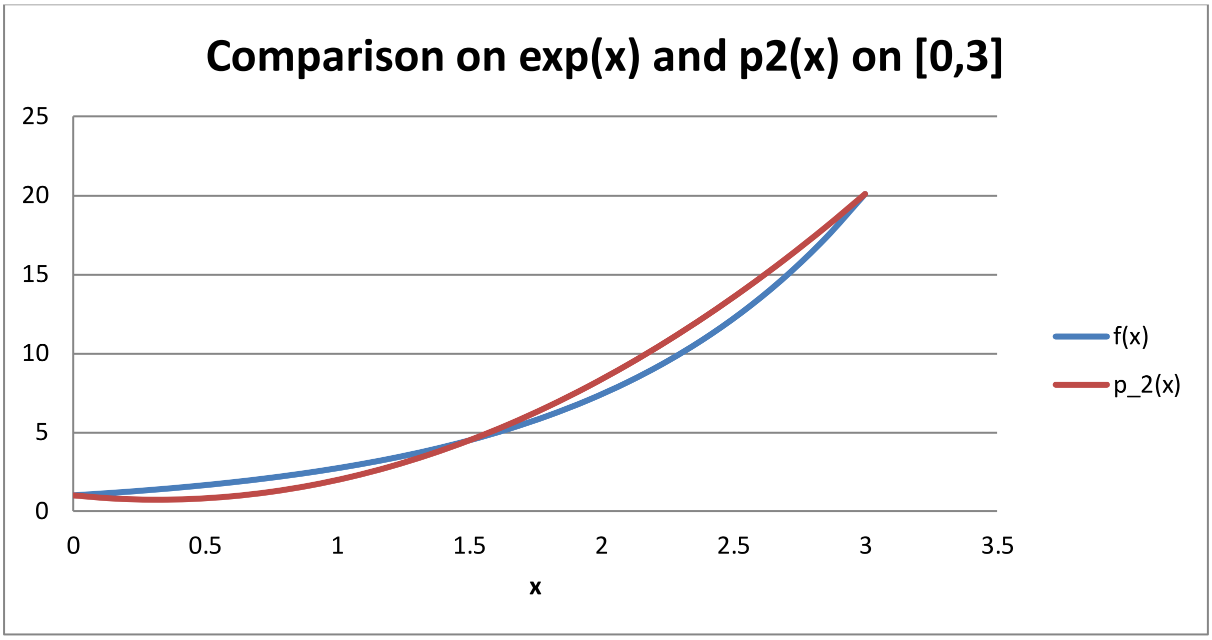
However, from a computational point of view, the layout adopted in the tables above demonstrates that the original column can be repeatedly overwritten from the diagonal downward. The final column of values are the ones used in the divided difference polynomial equation. This is economical in terms of use of computer memory.

Further Examples

In the following examples Newton's divided difference method for approximating the function e^x on the domain [0,3] using equidistant interpolation points. These examples are also illustrated on an Excel spreadsheet[2]. Firstly, let us approximate the function by a quadratic. The following table shows the value of the function at the three equidistant points that are used for interpolation.

x_i	$f(x_i)$
0	1
1.5	4.481689
3	20.08554

The following graph compares the function e^x with the quadratic interpolant.

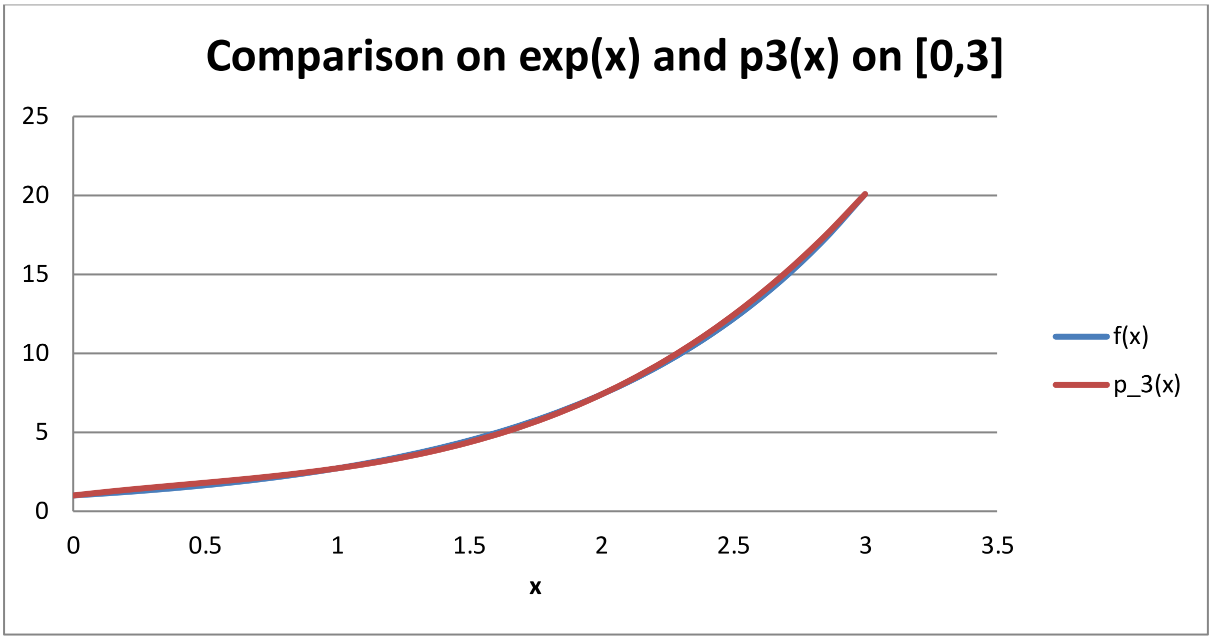


Firstly, let us approximate the function by a cubic. The following table shows the value of the function at the four equidistant points that are used for interpolation.

x_i	$f(x_i)$
0	1
1	2.718282
2	7.389056
3	20.08554

[2] Polynomial Interpolation (Spreadsheet)

The following graph compares the function e^x with the cubic interpolant.



It may be thought that the higher the degree of the interpolating polynomial, the more accurate the interpolant is. This is not the case, unless we choose the interpolation points carefully. For example the following graph interpolates the function $f(x) = \frac{1}{1+x^2}$ in [-3,3] with seven equidistant interpolation points.